

Derive exact BER expression for a wireless system:

$$\text{Average BER} = \int_0^{\infty} Q(\sqrt{a^2 \text{SNR}}) 2a e^{-a^2} da$$

$$\text{SNR} = \mu$$

$$= \int_0^{\infty} Q(\sqrt{a^2 \mu}) 2a e^{-a^2} da$$

$$Q(v) = \int_v^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

$$BER = \underbrace{\int_0^\infty \int_{\sqrt{a^2 \mu}}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \cdot 2a e^{-a^2} da}_{\substack{\text{inner } t \\ \text{outer } a}}$$

$\frac{t}{a\sqrt{\mu}} = u \quad dt = a\sqrt{\mu} du$

$$BER = \int_0^\infty \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2 a^2 \mu}{2}} \cdot a\sqrt{\mu} du \cdot 2a e^{-a^2} da$$

$$\text{BER} = \sqrt{\mu} \int_0^{\infty} \int_1^{\infty} \frac{2a^2}{\sqrt{2\pi}} e^{-\frac{a^2}{2}(2+\mu u^2)} du \cdot da$$

$$= \sqrt{\mu} \int_1^{\infty} \int_0^{\infty} \frac{2a^2}{\sqrt{2\pi}} e^{-\frac{a^2}{2}(2+\mu u^2)} da \, du$$

inner a

outer, μ .

$$\int_0^{\infty} \frac{2a^2}{\sqrt{2\pi}} e^{-a^2/2(2+\mu u^2)} da = \sigma^3 = \left(\frac{1}{2+\mu u^2} \right)^{3/2}$$

For a Gaussian

$$\int_{-\infty}^{\infty} \frac{t^2}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt = \sigma^2$$

$$\Rightarrow 2 \int_0^{\infty} \frac{t^2}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{t^2}{2\sigma^2}} dt = \sigma^2$$

$$\Rightarrow \int_0^{\infty} \frac{2t^2}{\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} dt = \sigma^3$$

$$\sigma^2 = \frac{1}{2+\mu u^2}$$

$$\frac{1}{\sigma^2} = 2+\mu u^2$$

Inner integral wrto $a = \left(\frac{1}{2 + \mu u^2} \right)^{3/2}$

BER = outer integral

$$= \sqrt{\mu} \int_0^{\infty} \left(\frac{1}{2 + \mu u^2} \right)^{3/2} du$$

$$du = \sqrt{\frac{2}{\mu}} \sec^2 \theta d\theta$$

$$u = \sqrt{\frac{2}{\mu}} \tan \theta$$

$$u = 1$$

$$\theta = \tan^{-1} \sqrt{\frac{\mu}{2}}$$

$$u = \infty$$

$$\theta = \tan^{-1} \infty = \frac{\pi}{2}$$

$$= \int_{\tan^{-1} \sqrt{\frac{\mu}{2}}}^{\pi/2} \left(\frac{1}{2 + 2 \tan^2 \theta} \right)^{3/2} \sqrt{\frac{2}{\mu}} \sec^2 \theta d\theta$$

$$BER = \sqrt{\mu} \int_{\tan^{-1} \sqrt{\frac{\mu}{2}}}^{\pi/2} \left(\frac{1}{2 + 2 \tan^2 \theta} \right)^{3/2} \sqrt{\frac{2}{\mu}} \sec^2 \theta d\theta$$

$$= \int_{\tan^{-1} \sqrt{\frac{\mu}{2}}}^{\pi/2} \frac{\sqrt{2}}{2^{3/2}} \cdot \frac{1}{\sec^3 \theta} \cdot \sec^2 \theta d\theta$$

$$= \frac{1}{2} \int_{\tan^{-1} \sqrt{\frac{\mu}{2}}}^{\pi/2} \cos \theta d\theta = \frac{1}{2} \sin \theta \Big|_{\tan^{-1} \sqrt{\frac{\mu}{2}}}^{\pi/2}$$

$$BER = \frac{1}{2} \left(1 - \sin \tan^{-1} \sqrt{\frac{\mu}{2}} \right)$$

$$\sin \theta = \sqrt{\frac{\tan^2 \theta}{1 + \tan^2 \theta}}$$

$$\sin \tan^{-1} \sqrt{\frac{\mu}{2}} = \sqrt{\frac{\tan^2 \tan^{-1} \sqrt{\frac{\mu}{2}}}{1 + \tan^2 \tan^{-1} \sqrt{\frac{\mu}{2}}}}$$

$$= \sqrt{\frac{\mu/2}{1 + \mu/2}} = \sqrt{\frac{\mu}{2 + \mu}}$$

$$BER = \frac{1}{2} \left(1 - \sqrt{\frac{SNR}{2 + SNR}} \right)$$

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Average BER
for BPSK in Rayleigh
Fading wireless system.